Second quantization of open systems using quasinormal modes

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The second quantization of a scalar field in an open cavity is formulated, from first principles, in terms of the quasinormal modes (QNMs), which are the eigensolutions of the evolution equation that decay exponentially in time as energy leaks to the outside. For any amount of damping, this formulation provides an exact description involving the cavity degrees of freedom only, with the outside acting as a (thermal or driven) source. Thermal correlation functions and cavity Feynman propagators are thus expressed in terms of the QNMs, labeled by a discrete index rather than a continuous momentum. Single-resonance domination of the density of states and the spontaneous decay rate is then given a proper foundation. This is an essential first step towards the application of QNMs to cavity quantum electrodynamics phenomena, to be reported elsewhere. [S1063-651X(98)14209-5]

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I. INTRODUCTION

In this paper we are concerned with quantum fields in *open* cavities—the obvious example of ultimate interest would be electromagnetic fields in optical cavities [1] and the associated problem of cavity quantum electrodynamics (CQED). Such systems are open because energy leaks to the outside (e.g., via output coupling), and as dissipative systems cannot normally be quantized on their own [2]; rather, one must also consider the bath into which energy escapes, so that the total universe is conservative. Thus such cavities, say, of linear dimension a, can be embedded in a universe of dimension $\Lambda\to\infty$. One can then quantize on the modes of the universe, which are labeled by a wave number p spaced by $\Delta p\sim\pi/\Lambda\to 0$. The field quanta are then destroyed or created by operators $a(p)$ and $a^\dagger(p)$, and higher-order processes involve integrals $\int dp$ over internal loops.

On the other hand, these cavities often have a very small amount of leakage, characterized by a parameter $\epsilon=Q^{-1}\ll 1$, where the quality factor of the cavity can be as high as $Q\sim 10^6$ or more. If this is the case, the intuition developed from a *closed* cavity, also of length a, should be relevant. A closed cavity is a conservative system, with normal modes (NMs) labeled by a *discrete* index $j=1,2,\ldots$, where the wave number is $p_j\sim j\pi/a$, $\Delta p\sim\pi/a$. Field quanta in such a closed cavity are destroyed and created by operators a_j and $a_j^\dagger$, and higher-order processes involve discrete sums Σ_j. Can quantum fields in an *open* cavity be described in a similar way—in terms of discrete modes and the corresponding operators? If this is possible, computations will be simplified and will correspond to physical intuition, with each term j associated with a cavity ''mode.'' The connection with the limit of a closed cavity ($\epsilon\to 0$) would also become manifest.

Quantization of a closed system relies on its NMs; the counterparts in an open system are the quasinormal modes (QNMs), which are again factorized solutions,

$$\phi(x,t)=f_j(x)e^{-i\omega_j t}, \tag{1.1}$$

where Im $\omega_j<0$ because of the loss of energy. Each QNM corresponds to a resonance, with a width $\gamma_j=|\mathrm{Im}\,\omega_j|$. The purpose of this paper, in short, is to develop a formalism whereby field quantization can be implemented in terms of these QNMs, and to define and study operators $a_j, a_j^\dagger$ for these modes. Specifically, one wishes to express field correlation functions, Feynman propagators, and other quantities in terms of the QNMs. The dissipative nature of the system is then contained in the QNMs themselves.

The advantages for CQED would be obvious. The simplest phenomenon to which such a formalism can apply is the well-known enhancement (or suppression) of spontaneous decay rates when the emitted radiation falls on (or between) resonances [3]. Resonance domination of these processes has been discussed starting with the heuristic argument due to Purcell [4]. He proposed that the Fermi golden rule should be generalized: the density of states per unit volume, instead of the usual $d_0(\omega)=\omega^2/(\pi^2c^3)$ for vacuum (where c is the velocity of light), is to be replaced by $d(\omega)\sim D/(2\gamma V)$ for a D-fold degenerate QNM of width γ in a cavity of volume V. This leads to an enhancement factor of $K=d/d_0\sim(1/8\pi)DQ(\lambda^3/V)$ for spontaneous emission on resonance, where λ is the wavelength of light emitted and Q is the quality factor of the cavity. The essence of this argument is that each resonance counts as one state, i.e., in a suitable sense it carries unit weight. While intuitively plausible, this statement is difficult to justify formally—since the entire concept of a state, i.e., an NM, falls apart in an open system. However, this argument, and its many variants and extensions, would find natural expression in a framework that quantizes on the QNMs, and we shall in particular show explicitly below that each resonance carries unit weight.

In Sec. II, the QNM expansion of *classical* fields outgoing from a cavity is reviewed. The classical results may be organized into two levels. First, under conditions to be specified, the Green's function G can be expanded in terms of QNMs. Second, one can try to expand the outgoing classical fields ϕ in terms of QNMs, and to establish a linear space structure similar to that for conservative systems. In order to do so, it turns out to be necessary to make use of a two-component formalism, dealing with ϕ and the conjugate momentum $\hat{\phi}$ at

the same time. The linear space formalism is more elegant, but in its simplest form is limited to only one dimension (1D).

The quantum formalism can likewise be approached in two ways. In the first, which we shall call the Green's-function method (Sec. III), one focuses on c-number correlation functions and propagators without explicitly expanding the field operator ϕ in terms of QNMs. The key idea is that the retarded propagator G^{R} of the quantum theory is exactly the same as the classical Green's function G, and the latter has a QNM expansion. Once G^{R} is obtained, it is straightforward to derive a QNM expansion for the correlation function F as well.

In the second, which we shall call the field expansion method, one tries to establish an expansion of the quantum field ϕ, in parallel with the linear space structure established for classical fields. This allows us to interpret the expansion coefficients a_j and $a_j^\dagger$, roughly speaking, as generalized annihilation and creation operators for the discrete QNMs j. However, before doing so, it has to be recognized that quantum fields cannot be constrained by the outgoing wave condition—for the simple reason that zero-point (and thermal) fluctuations must contain an incoming component. Thus, the first step in developing this method, presented in Sec. IV, is to generalize the field expansion to handle *incoming* waves as well. With this generalization, one can then subject the fields to canonical quantization in Sec. V. This is done by starting with the universe, a closed Hermitian system for which the quantization is unambiguously defined. Then, in parallel with the usual removal of bath oscillators [2], the outside degrees of freedom are eliminated from the equations of motion [5]. The results will be equations of motion and commutation relations for the *discrete* operators a_j and $a_j^\dagger$, in which the effects of the outside bath are clearly displayed: the loss of energy of each mode by leakage, and the pumping of each mode by the quantum or thermal fluctuations from the outside.

The formalism is then used to evaluate the correlation function F in Sec. VI, and the results are compared with those obtained from the Green's-function method. Interestingly, the results appear to be different—those derived from the Green's-function method contain a single sum Σ_j over the QNMs, while the field expansion method yields a double sum Σ_{jk} with off-diagonal terms. The two are, however, shown to be equal through an identity on G^{R}. Recalling that the expansion of the classical field is unique only when the second component $\hat{\phi}$ is considered at the same time, we next show that the expansion of the correlation function F is also unique if we consider $\hat{\phi}$ as well, giving the nondiagonal form. The density of states d, which is intimately related to the correlation function, is also expressed in terms of QNMs; in particular, it is shown that up to corrections of $O(Q^{-1})$, each resonance carries unit weight in the density of states.

The results on the correlation function are then used, in Sec. VII, to evaluate and discuss the Feynman propagator G^{F}, which is the fundamental building block for CQED. Again, equivalent diagonal and nondiagonal forms are obtained. Particular attention is paid to the equal-space propagator $\tilde{G}^{\mathrm{F}}(x,x,\omega)$, whose imaginary part is related to the lifetime of an excited atom placed at x. This quantity is discussed in the approximation of domination by a single resonance, providing justification for Purcell's heuristic argument [4] on the enhancement of spontaneous decay rates. The advantage of using the nondiagonal expression is again emphasized.

In Sec. VIII, a very simple example is studied explicitly, and its correlation function and energy density are expressed in terms of a sum over QNM contributions.

Some final remarks are then given in Sec. IX. We stress that this paper is concerned entirely with *free* fields, either as a model of the free electromagnetic field in an optical cavity, or as the zeroth-order building blocks in an interacting theory, e.g., the propagators as ingredients in higher-order Feynman diagrams. The development of the interacting theory and its application to CQED phenomena will be given elsewhere [6]. A partial account of the present theory has been given in [7].

II. CLASSICAL FIELDS

In this section, we summarize the QNM expansion for classical fields. In this paper, we deal with scalar fields in 1D only.

For closed, linear systems, eigenfunction expansions, based on the eigenfunctions or NMs of their evolution operators, are a tool of vital importance in theoretical physics. However, open systems are not directly amenable to an NM analysis. Examples of open systems include optical cavities [1], and finite regions of space near astrophysical objects, from which gravitational waves can escape [8,9]. In these systems, any initial state decays in time, so stationary NMs do not exist. As the simplest example, we shall be concerned with the real scalar wave equation in one space dimension,

$$\rho(x)\partial_t^2\phi=\partial_x^2\phi, \tag{2.1}$$

studied in a ''cavity'' $0\leqslant x\leqslant a$, with the nodal boundary condition

$$\phi(x=0,t)=0 \tag{2.2}$$

at one end but with the outgoing one

$$\phi'(a^+,t)=-\dot{\phi}(a,t) \tag{2.3}$$

at the other. The latter condition states that, just outside the cavity boundary, the field $\phi(x,t)$ is an outgoing wave $\phi(x-t)$; the condition is specified at a^+ because, as we shall see below, one is often concerned with models in which there is a singularity in $\rho(x)$ at $x=a$, leading to a possible discontinuity in $\phi'(x)$ [10]. The boundary condition (2.3) turns the cavity into a dissipative system that is leaky but not absorptive. The model (2.1) has been widely used as the scalar model of electromagnetism in an optical cavity [1]. More physically, the 1D nature is realized in Fabry-Perot cavities with lengths much smaller than the lateral dimensions, and the scalar field model is rigorously applicable to the transverse electric sector.

For the open system (2.1)–(2.3), the eigensolutions, labeled by an index j, have the form (1.1), with the QNMs or cavity resonances f_j satisfying

$$[\partial_x^2+\rho(x)\omega_j^2]f_j=0 \tag{2.4}$$

and the boundary conditions (2.2), (2.3) translating into

$$f_j(0)=0, \quad f_j'(a^+)=i\omega_j f_j(a). \tag{2.5}$$

It is easily verified that Im $\omega_j<0$, so that the solution (1.1) is indeed decaying in time. Furthermore, the frequencies ω_j, which we suppose to be ordered according to increasing real parts, are spaced by $\Delta\omega\sim\pi/a$, approximately as for a conservative system of size a. With the possible exception of modes with Re $\omega_j=0$, the QNMs always occur in pairs with $\omega_{-j}=-\omega_j^*$, and one can choose $f_{-j}=f_j^*$. While the field ϕ is real, the eigenvalues and eigenfunctions are complex; this is the reason for the pairing of modes.

The usual formalism concerning eigenfunction expansions relies on the hermiticity of the evolution operator, which only holds in the conservative case, and therefore breaks down for open systems. One possible resolution is to embed the cavity into a ''universe'' $0\leqslant x\leqslant\Lambda$ with a nodal condition at $x=\Lambda\to\infty$, and study its NMs—the modes of the universe. Namely, the system (2.1)–(2.3) is the restriction to $x\leqslant a$ of the problem (2.1) on the half-line $0\leqslant x<\infty$, if one sets

$$\rho(x>a)\equiv 1 \tag{2.6}$$

and with the extension of the initial conditions to the ''outside'' $x>a$ obeying $\phi'(x>a,t=0)=-\dot{\phi}(x>a,t=0)$. However, this has the obvious disadvantage of having to work with a continuum of states (spaced by $\Delta\omega\sim\pi/\Lambda\to 0$) as opposed to the discrete set of eigenfunctions in the conservative case. Besides, the closed system of Eqs. (2.1)–(2.3) shows that even in the presence of dissipation the time evolution of the cavity can be studied *without* explicit reference to the outside, which is the principal goal of the program of second quantization of the open system.

Previous work (see [11–13] and references therein) has established that, in spite of the lack of hermiticity in the conventional sense, an eigenfunction expansion for outgoing waves in classical open wave systems can be formulated in terms of the cavity degrees of freedom only, overcoming the disadvantages of the modes of the universe approach. The sufficient conditions for this QNM expansion are as follows. (a) The function $\rho(x)$ has at least a step discontinuity at $x=a$. This demarcates a well-defined cavity region. (b) The function $\rho(x)$ has no tail outside the cavity, i.e., $\rho(x>a)\equiv 1$. This condition ensures that the outside does not reflect outgoing waves back into the cavity, enabling the complete elimination of the environment from the equations of motion. These conditions are satisfied for optical cavities bounded from extended vacuum by a sharp material interface. Under these conditions, the eigenfunction expansion is exact for any amount of dissipation.

The completeness of the QNMs can be pursued at two levels. First, one shows that the retarded Green's function of the system has the representation

$$G(x,y,t)=\sum_j \frac{f_j(x)f_j(y)}{2i\omega_j}e^{-i\omega_j t} \tag{2.7}$$

for $0\leqslant x,y\leqslant a$, and $t\geqslant 0$, where the f_j's are normalized according to Eq. (2.11) below. Thus, the dynamics is contained entirely in the QNMs, leading to a simple method of obtaining the retarded propagators and quantum correlation functions, as sketched in Sec. III.

Second, realizing that the wave equation (2.1), like any classical Hamiltonian problem, requires both position and momentum to be specified as initial data, one introduces function *pairs* $\boldsymbol{\phi}=(\phi,\hat{\phi})^{\mathrm{T}}$ with the conjugate momentum $\hat{\phi}\equiv\rho\dot{\phi}$, so that for eigenfunctions $\boldsymbol{f}_j=(f_j,-i\rho\omega_j f_j)^{\mathrm{T}}$. The space of all function pairs (in general allowed to be complex) satisfying the boundary conditions (2.2) and (2.3) will be denoted as Γ—the space of outgoing waves.

Using these pairs, one can prove that the time evolution generated by Eq. (2.7) can be recast in the form [14]

$$\boldsymbol{\phi}(t)=\sum_j a_j(t)\boldsymbol{f}_j, \tag{2.8}$$

where the expansion coefficients are given by

$$a_j(t)=\frac{1}{2\omega_j}\langle \boldsymbol{f}_j,\boldsymbol{\phi}(t)\rangle \tag{2.9}$$

with $a_j(t)=a_j(0)e^{-i\omega_j t}$ and the *bilinear scalar product* for $\boldsymbol{\zeta},\boldsymbol{\chi}\in\Gamma$,

$$\langle\boldsymbol{\zeta},\boldsymbol{\chi}\rangle=i\left\{\int_0^{a^+}dx[\zeta(x)\hat{\chi}(x)+\hat{\zeta}(x)\chi(x)]+\zeta(a)\chi(a)\right\}. \tag{2.10}$$

By simply letting $t\downarrow 0$ in Eq. (2.8) one arrives at a *two-component expansion* for an arbitrary real $\boldsymbol{\phi}\in\Gamma$ [15,16]. This expansion makes the completeness of the QNMs manifest. The normalization used in Eqs. (2.7)–(2.9) can be concisely expressed as

$$\langle \boldsymbol{f}_j,\boldsymbol{f}_j\rangle=2\omega_j. \tag{2.11}$$

It is seen that Eq. (2.11) in general is not real, underlining the difference between the product (2.10) and a conventional one involving complex conjugation. The fact that Eq. (2.11) is bilinear also serves to establish a phase convention for the wave functions.

Upon introducing the two-component evolution operator

$$\boldsymbol{H}=i\begin{pmatrix}0 & \rho(x)^{-1}\\ \partial_x^2 & 0\end{pmatrix}, \tag{2.12}$$

the cavity evolution (2.1) can be written as $i\partial_t\boldsymbol{\phi}=\boldsymbol{H}\boldsymbol{\phi}$, in striking analogy with quantum mechanics. In this notation, the definition (2.4) of f_j takes the form $\boldsymbol{H}\boldsymbol{f}_j=\omega_j\boldsymbol{f}_j$. The operator $\boldsymbol{H}$ can be shown to be symmetric with respect to the form (2.10), i.e.,

$$\langle\boldsymbol{\zeta},\boldsymbol{H}\boldsymbol{\chi}\rangle=\langle\boldsymbol{\chi},\boldsymbol{H}\boldsymbol{\zeta}\rangle \tag{2.13}$$

for any $\boldsymbol{\zeta},\boldsymbol{\chi}\in\Gamma$. This analog of Hermiticity holds even though the system is not conservative. The symmetry of $\boldsymbol{H}$ yields the orthogonality relation

$$\langle \boldsymbol{f}_j,\boldsymbol{f}_k\rangle=0 \quad \text{for } \omega_j\neq\omega_k \tag{2.14}$$

in an immediate transcription of the usual proof, leading to the uniqueness of the expansion. Incidentally, an expansion such as Eq. (2.8) but involving the first component alone would not be unique.

Instead of its present formulation as an orthogonal expansion involving a generalized bilinear scalar product, the series (2.8) can also be regarded as a biorthogonal expansion in terms of the standard inner product [17]. The power of this latter, slightly more involved method shows when several QNMs merge [18], a possibility not considered in this paper.

It is appropriate to contrast the two methods of approaching the classical theory, since they respectively underpin the two methods for dealing with the quantum case. The expansion of the Green's function is easy to derive, and is readily generalized to higher dimensions; however, in itself it does not lead to a *unique* expansion of the field, nor to concepts of orthogonality. The two-component approach based on Eq. (2.8) is more elegant, exhibits a deeper resemblance to conservative systems, and most importantly leads to a *unique* expansion in terms of orthogonal functions. The two-component expansion can in principle be generalized to higher dimensions by treating each angular momentum sector l as a 1D radial problem [19], but the degree of complexity increases with l. Thus each method has its own merits; both will be pursued below, and the results compared.

III. GREEN'S-FUNCTION METHOD

The quantum mechanics of the cavity plus outside is specified by the Hamiltonian

$$H=\int_0^\infty dx\ h(x)=\int_0^\infty dx\left[\frac{\hat{\phi}^2}{2\rho}+\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2\right] \tag{3.1}$$

together with the canonical equal-time commutation relation,

$$[\phi(x),\hat{\phi}(y)]=i\delta(x-y). \tag{3.2}$$

Time evolution is then generated by means of the Heisenberg equation $\dot{A}=i[H,A]$ for an arbitrary operator A. However, instead of the equations of motion for the quantum operators, in this section we focus first on the retarded propagator

$$G^{\mathrm{R}}(x,y,t)=-i\theta(t)\langle[\phi(x,t),\phi(y)]\rangle, \tag{3.3}$$

in which ϕ is of course to be regarded as an operator, and $\langle\cdots\rangle$ denotes the expectation value at a finite temperature $T=1/\beta$; throughout we take $\hbar=k_{\mathrm{B}}=1$.

The central idea is that this propagator defined in terms of the quantum fields can be evaluated without explicitly introducing an expansion for the field operators ϕ, by simply noticing that $G^{\mathrm{R}}(x,y,t)$ is exactly the same as the Green's function G of the classical wave equation [20], which has the expression (2.7) in terms of QNMs. This relationship between G^{R} and G follows from the commutation relation (3.2).

In terms of G^{R}, it is straightforward to compute the equilibrium correlation function

$$F(x,y,t)\equiv\langle\phi(x,t)\phi(y)\rangle. \tag{3.4}$$

We shall devote attention to F, because the physical quantities of interest in quantum field theory can often be formulated in terms of correlation functions, at either zero or finite temperatures. For example, the Casimir force is merely the vacuum expectation value of the electromagnetic stress tensor, which is an equal-time equal-space correlation function of two field operators. The spontaneous decay rate of an atom in an excited state is, in the golden-rule approximation, related to the correlation function of two electric-field operators.

Since the correlation function is related to the retarded propagator [20], one gets

$$\tilde{F}(x,y,\omega)=\frac{-2}{1-e^{-\beta\omega}}\ \mathrm{Im}\ \tilde{G}^{\mathrm{R}}(x,y,\omega) \tag{3.5}$$

$$=\frac{i\omega}{1-e^{-\beta\omega}}\sum_j\ \frac{f_j(x)f_j(y)}{\omega_j(\omega^2-\omega_j^2)}. \tag{3.6}$$

The real-time correlator can be obtained from Eq. (3.6) by contour integration, yielding

$$F(x,y,t)=\sum_j\ \frac{f_j(x)f_j(y)}{2\omega_j(1-e^{-\beta\omega_j})}e^{-\beta\omega_j\theta(-t)-i\omega_j|t|} + \sum_{m=1}^{\infty}\frac{e^{-\mu_m|t|}}{\beta}[\tilde{G}^{\mathrm{R}}(x,y,-i\mu_m)-\tilde{G}^{\mathrm{R}}(x,y,i\mu_m)]. \tag{3.7}$$

The first term in this formula is due to the QNM poles in $\tilde{F}(x,y,\omega)$; the second term, which has no counterpart in $G^{\mathrm{R}}(x,y,t)$, is caused by the Matsubara poles in $\tilde{F}(x,y,\omega)$ at frequencies $\mu_m\equiv 2\pi mT$. This very simple derivation has the advantage that it goes through in situations where the two-component formalism may be more complicated.

In principle, physical quantities can be expressed in terms of F—bilinear quantities (such as the energy density) as linear combinations of F and its derivatives, and other quantities involving products of F's. For example, the energy density is

$$\langle h(x)\rangle=\tfrac{1}{2}[-\rho(x)\partial_t^2+\partial_x\partial_y]F(x,y,t)|_{x=y,t=0}. \tag{3.8}$$

However, this quantity is divergent. Subtracting off the zero point, we consider

$$U(x,T)=\langle h(x)\rangle-\langle h(x)\rangle_{T=0} =\tfrac{1}{2}[-\rho(x)\partial_t^2+\partial_x\partial_y]F_{\mathrm{S}}(x,y,t)|_{x=y,t=0} \tag{3.9}$$

in terms of the subtracted correlation function $F_{\mathrm{S}}\equiv F-F_0$, where $F_0=\lim_{\beta\to\infty}F$. The limit $\beta\to\infty$ is best taken in Eq. (3.6) prior to Fourier inversion. We further make use of the expansion for $\tilde{G}^{\mathrm{R}}$ to get

$$F_{\mathrm{S}}(x,y,t)=\sum_j\ \frac{f_j(x)f_j(y)}{\omega_j}C_j(t), \tag{3.10}$$

$$C_j(t)=\frac{e^{-i\omega_j t}}{2}\left[\frac{1}{e^{\beta\omega_j}-1}+\theta(-\mathrm{Re}\ \omega_j)\right]+\frac{i}{\beta}\sum_{m=1}^{\infty}\frac{\mu_m e^{-\mu_m t}}{\mu_m^2+\omega_j^2}-\frac{i}{4\pi}e^{i\omega_j t}E_1(i\omega_j t)-\frac{i}{4\pi}e^{-i\omega_j t}E_1(-i\omega_j t), \tag{3.11}$$

where this and subsequent formulas for F_S are written only for $t>0$. Here, $E_1(z)$ is the exponential integral function [21]

$$E_1(z)=\int_z^{+\infty}\frac{e^{-u}}{u}\,du, \tag{3.12}$$

in which the integration contour is defined not to pass through the origin and the negative real axis; on that semiaxis, the function is defined as the principal value. We further define $\theta(0)\equiv\frac{1}{2}$.

Alternatively, for greater formal similarity to the conservative case, Eq. (3.11) can be rewritten as

$$F_S(x,y,t)=\sum_{\mathrm{Re}\,\omega_j\geqslant 0}{}' \ 2\,\mathrm{Re}\left[\frac{f_j(x)f_j(y)}{\omega_j}\,\tilde{C}_j(t)\right], \tag{3.13}$$

$$\tilde{C}_j(t)=\frac{e^{-i\omega_j t}}{2(e^{\beta\omega_j}-1)}+\frac{i}{\beta}\sum_{m=1}^{\infty}\frac{\mu_m e^{-\mu_m t}}{\mu_m^2+\omega_j^2}-\frac{i}{4\pi}e^{i\omega_j t}E_1(i\omega_j t)-\frac{i}{4\pi}e^{-i\omega_j t}E_1(-i\omega_j t). \tag{3.14}$$

The prime on the sum in Eq. (3.13) signifies that terms with $\mathrm{Re}\ \omega_j=0$ are to be taken with weight $\frac{1}{2}$.

The actual evaluation of $U(x,T)$ needs some care in the $j\to\infty$ part of the sums. These details, and the very similar problem for the calculation of the Casimir force, will be given elsewhere.

IV. INCOMING WAVES

The expansion of a classical field sketched in Sec. II is restricted to outgoing waves, i.e., to $\boldsymbol{\phi}\in\Gamma$, satisfying Eq. (2.3). In preparing the ground for the expansion of a *quantum* field, it is necessary to remove this restriction, for the simple reason that the zero-point quantum fluctuations will inevitably contain incoming waves as well. Moreover, one would wish that the ensuing theory should be applicable to situations where there is an incoming pump field.

Thus, we study the wave equation (2.1) for the system together with the outside ''bath,'' i.e., on the half-line $x>0$, with $\rho(x)$ satisfying Eq. (2.6) and the boundary condition (2.2). The initial conditions are now arbitrary and accordingly the outgoing boundary condition (2.3) is abandoned, i.e., the restriction of $\boldsymbol{\phi}$ to the cavity need not lie in Γ. For the outside $x>a$ [where $\rho(x)=1$] the initial data are decomposed as

$$\boldsymbol{\phi}(x>a,0)=\boldsymbol{\phi}_{\mathrm{in}}+\boldsymbol{\phi}_{\mathrm{out}}, \tag{4.1}$$

with $\boldsymbol{\phi}_{\mathrm{in}}$ satisfying the incoming wave condition $\phi'_{\mathrm{in}}=\hat{\phi}_{\mathrm{in}}$, while $\phi'_{\mathrm{out}}=-\hat{\phi}_{\mathrm{out}}$. For the cavity subsystem this decomposition leads to the boundary condition

$$\phi'(a^+,t)+\hat{\phi}(a^+,t)=2\,\hat{\phi}_{\mathrm{in}}(a+t)\equiv b(t), \tag{4.2}$$

where the driving force b [see Eq. (4.4) below for its name], being determined by the initial data, is supposedly a known function (at least in a statistical sense) that characterizes the waves incoming from the outside. Inside the cavity, the field is then expanded in terms of QNMs by Eq. (2.8) with

$$a_j=\frac{1}{2\omega_j}\langle\boldsymbol{f}_j,\boldsymbol{\phi}\rangle=\frac{i}{2\omega_j}\left\{\int_0^{a^+}dx\ f_j(x)[\hat{\phi}(x)-i\rho(x)\omega_j\phi(x)]+f_j(a)\phi(a)\right\}. \tag{4.3}$$

That is, we *retain* the expansion formula and the inner product definition and notation even though $\boldsymbol{\phi}\notin\Gamma$. As a consequence, the sum in Eq. (2.8) will in general not converge to $\hat{\phi}$ at $x=a$, the point where the boundary condition is imposed. Nevertheless, the sum for the first component converges to ϕ everywhere, while the sum for the second component converges to $\hat{\phi}$ everywhere except at $x=a$ [22]. (This is most easily appreciated by noticing that upon changing $\hat{\phi}$ at just one point, the resultant wave function can be made to lie in Γ. In the Hilbert-space setting of Refs. [17,18], Γ is a dense vector subspace, not a closed subspace of codimension 1; this clarifies further why there is no extra degree of freedom associated with incoming waves.) This flaw on a set of measure zero does not lead to problems, however, for the projection formula (4.3) renders the coefficients $a_j(t)$ well defined irrespective of the convergence of the series (2.8).

The equation of motion for a_j will now be derived. By differentiating Eq. (4.3) with respect to time, and then integrating by parts, one obtains

$$\dot{a}_j(t)+i\omega_j a_j(t)=\frac{i}{2\omega_j}f_j(a)b(t). \tag{4.4}$$

In contrast to the case of pure outgoing waves, there is now an extra term on the right-hand side: each QNM is driven by the ''force'' $b(t)$, and at the same time decays because of $\mathrm{Im}\ \omega_j$. The coupling to the ''force'' is determined by the surface value of the QNM wave function $f_j(a)$. This equation of motion will survive quantization, to be carried out in the next section.

V. FIELD EXPANSION METHOD

A second approach to second quantization proceeds more explicitly by first promoting ϕ and $\hat{\phi}$ to operators [23]. These fields may be regarded as operators for the entire ''universe,'' which is a conservative system to which canonical quantization can be applied. The same projection formula (4.3) as in the classical case now defines the a_j's as Hilbert

space operators, obeying the equation of motion (4.4).

The crucial point is that the field commutation relation (3.2) and the projection formula (4.3) now lead directly to commutators for these coefficients, viz.,

$$[a_j,a_k]=\frac{1}{4\omega_j\omega_k}[\langle \boldsymbol{f}_j,\boldsymbol{\phi}\rangle,\langle \boldsymbol{f}_k,\boldsymbol{\phi}\rangle]$$
$$=-\frac{1}{4\omega_j\omega_k}\left\{\int_0^{a^+} dx\,dy\;(f_j(x)\hat{f}_k(y)[\hat{\phi}(x),\phi(y)]+\hat{f}_j(x)f_k(y)[\phi(x),\hat{\phi}(y)])+\int_0^{a^+} dx\;f_j(x)f_k(a)[\hat{\phi}(x),\phi(a)]+\int_0^{a^+} dy\;f_j(a)f_k(y)[\phi(a),\hat{\phi}(y)]\right\}. \tag{5.1}$$

In these equations, ϕ and $\hat{\phi}$ are q numbers, while f_j, f_k are c numbers. The two surface terms on the last line cancel as long as the δ function at the boundary of the integration interval is interpreted consistently. In the first line, the commutation relation (3.2) gives $\delta(x-y)$ and cancels one integration. One is then left with

$$[a_j,a_k]=\frac{\omega_k-\omega_j}{4\omega_j\omega_k}\int_0^{a^+} dx\;\rho(x)f_j(x)f_k(x) \tag{5.2}$$

$$=\frac{i(\omega_j-\omega_k)f_j(a)f_k(a)}{4\omega_j\omega_k(\omega_j+\omega_k)}, \tag{5.3}$$

where the second form follows from the first by means of the orthogonality relation (2.14), and will be useful later for comparison with results from Sec. VI.

The linear-space structure for open systems involves projections based on the generalized inner product (2.10), which is bilinear rather than linear in one vector and conjugate linear in the other; thus the expression in Eq. (5.2) involves an integral over $f_j(x)f_k(x)$ without complex conjugation. However, for the sake of a more transparent comparison with the conservative case, it is useful to rewrite these expressions by changing $j\mapsto -j$ and using $a_{-j}=a_j^\dagger$, $\omega_{-j}=-\omega_j^*$, and $f_{-j}=f_j^*$ to give

$$[a_j^\dagger,a_k]=-\frac{\omega_k+\omega_j^*}{4\omega_j^*\omega_k}\int_0^{a^+} dx\;\rho(x)f_j^*(x)f_k(x) \tag{5.4}$$

$$=-\frac{i(\omega_j^*+\omega_k)f_j^*(a)f_k(a)}{4\omega_j^*\omega_k(\omega_j^*-\omega_k)}. \tag{5.5}$$

The result in the form (5.4) reveals the conservative limit most clearly; in this limit the integral would simply be $\delta_{|j|,|k|}$.

Comparison of Eqs. (5.4) and (5.5) shows that

$$\frac{|f_j(a)|^2}{2|\mathrm{Im}\;\omega_j|}\to 1 \tag{5.6}$$

in the conservative limit. A more explicit proof is given in Appendix A.

The above commutators show that, if we define, for $j>0$,

$$\alpha_j=\sqrt{2\omega_j}a_j,$$
$$\alpha_j^\dagger=\sqrt{2\omega_j^*}a_{-j}, \tag{5.7}$$

then in the conservative limit these should reduce to the annihilation and creation operators, respectively [24]. Indeed, the QNM expansion (2.8) then takes the form

$$\begin{pmatrix}\phi\\ \hat{\phi}\end{pmatrix}=\sum_{j>0}\begin{pmatrix}(\alpha_j^\dagger+\alpha_j)/\sqrt{2\omega_j}\\ i\rho\sqrt{\omega_j/2}(\alpha_j^\dagger-\alpha_j)\end{pmatrix}f_j, \tag{5.8}$$

the standard NM field expansion for a closed cavity [16]. For finite damping, however, the operators a_j have mixed creation and annihilation character.

In short, we have established an expansion of the quantum field ϕ (and its conjugate momentum $\hat{\phi}$) in terms of operators a_j and $a_j^\dagger$, and then obtained equations of motion and commutation relations for the latter. This, in principle, completes the program of second quantization, and it remains to use these results to compute correlation functions and propagators, which we proceed to do in the following sections.

However, the deviation of the commutators (5.3) and (5.5) from the canonical form prevents the construction of a Fock space, as is the case for quantum dissipative systems in general [2].

VI. CORRELATION FUNCTIONS

The formalism derived in the last section for expanding the quantum field ϕ in terms of the operators a_j and $a_j^\dagger$ will be applied to the calculation of equilibrium correlation functions, yielding discrete representations for the cavity correlator F in the presence of dissipation. Section VI A contains the calculation of *F per se*, and Sec. VI B compares the results with those obtained from the Green's-function approach in Sec. III. In Sec. VI C we evaluate and discuss the density of states.

A. Field-field correlator

In Secs. VI to VIII we take the system to be in equilibrium. Then, the initial conditions for Eq. (4.4) are irrelevant and the dynamics are completely specified by the driving force b, i.e.,

$$a_j(t)=\frac{if_j(a)}{2\omega_j}\int_{-\infty}^{t} dt' e^{i\omega_j(t'-t)}b(t'). \tag{6.1}$$

The nonzero imaginary part of the ω_j renders the integral rapidly converging, in contrast to the conservative case. Fourier transforming and taking expectation values then lead to

$$\langle\tilde{a}_j(\omega)a_k\rangle=\frac{f_j(a)f_k(a)}{4\omega_j\omega_k(\omega_j-\omega)(\omega_k+\omega)}\langle\tilde{b}(\omega)b\rangle. \tag{6.2}$$

Since b is fully specified by the *in*coming waves from the *free* string $a<x<\infty$, it does not ''know'' about the cavity

$x \leqslant a$, so one can use the free infinite-string correlation function to calculate its spectral density from the definition (4.2) as

$$\begin{aligned}\langle \tilde{b}(\omega) b\rangle &= -(\partial_x - i\omega)^2 \langle \tilde{\phi}(x,\omega)\phi(y)\rangle_{\text{free}}\big|_{y=x} \\ &= -(\partial_x - i\omega)^2 \frac{\cos[\omega(x-y)]}{\omega(1-e^{-\beta\omega})}\bigg|_{y=x} \\ &= \frac{2\omega}{1-e^{-\beta\omega}}. \end{aligned} \tag{6.3}$$

For a simple check, antisymmetrize Eq. (6.2) in j and k and perform the inverse Fourier transform to reproduce Eq. (5.3) (for the expectation value of the commutator). Incidentally, by assuming other forms for $\langle \tilde{b}(\omega) b\rangle$, the theory accommodates various incoming pump fields.

Given the two-point function (6.3) for the driving force, it is straightforward to compute the two-point function for the response, namely the field-field correlation function inside the cavity. This now merely requires summation, that is, combination of Eqs. (2.8), (6.2), and (6.3) leads to

$$\tilde{F}(x,y,\omega) = \frac{\omega}{1-e^{-\beta\omega}} \times \sum_{jk} \frac{f_j(a) f_k(a)}{2\omega_j \omega_k (\omega_j - \omega)(\omega_k + \omega)} f_j(x) f_k(y). \tag{6.4}$$

The above derivation leads to a clear physical interpretation of the pole structure of Eq. (6.4) in the complex ω plane: the Matsubara poles at $\omega = i\mu_m = 2i\pi m T$ ($m \in \mathbf{Z}$) arise from the thermal character of the incoming noise, while the QNM poles correspond to cavity resonances excited by this noise.

B. Comparison of two forms for the correlation function

It will be noticed that we have obtained two different QNM expansions for $\tilde{F}$, namely, the double sum in Eq. (6.4) and the single sum in Eq. (3.6). We next prove their equivalence without invoking the QNM expansion of a quantum field.

To do so, we rely on the identity [25]

$$\tilde{G}^{\mathrm{R}}(x,y,\omega) - \tilde{G}^{\mathrm{R}}(x,y,-\omega) = \frac{2\omega}{i} \tilde{G}^{\mathrm{R}}(x,a,\omega) \tilde{G}^{\mathrm{R}}(y,a,-\omega) \tag{6.5}$$

for $x,y \leqslant a$. This identity, proved in Appendix B, has no nontrivial counterpart in closed, conservative systems. For an interpretation, notice that $\tilde{G}^{\mathrm{R}}(x,y,\omega) - \tilde{G}^{\mathrm{R}}(x,y,-\omega) \propto \mathrm{Im}\, \tilde{G}^{\mathrm{R}}(x,y,\omega)$ vanishes in the conservative limit and hence is a measure of dissipation, which the right-hand side states as taking place exclusively at the surface $x = a$.

Given this identity, the equivalence of the two expressions for $\tilde{F}$ follows simply by canceling the Bose prefactors in Eqs. (3.6) and (6.4) and comparing the result with the Fourier transform of Eq. (2.7).

Although the two forms are equivalent, each has its own attractive properties. The diagonal form (3.6) is simpler, while the nondiagonal form (6.4) is manifestly factorizable: $\tilde{F}(x,y,\omega) = A(\omega)\chi(x,\omega)\chi(y,-\omega)$ [26]. Anticipating a similar structure for Feynman propagators, the nondiagonal form permits a quantum in one mode j to propagate to another mode k, while the diagonal form implies that the mode index is ''conserved.''

The expansion of correlations involving ϕ alone is not unique, on account of the doubling of QNMs (j and $-j$) compared to NMs [12,13]. As discussed in Sec. II, it is more natural to consider $\boldsymbol{\phi} = (\phi, \hat{\phi})^{\mathrm{T}}$, which leads to a unique expansion. Thus we define a tensor field-field correlator,

$$\begin{aligned}\tilde{\mathsf{F}}(x,y,\omega) &\equiv \langle \tilde{\boldsymbol{\phi}}(x,\omega) \otimes \boldsymbol{\phi}(y)\rangle \\ &= \begin{pmatrix} \langle \phi(x,\omega)\phi(y)\rangle & \langle \phi(x,\omega)\hat{\phi}(y)\rangle \\ \langle \hat{\phi}(x,\omega)\phi(y)\rangle & \langle \hat{\phi}(x,\omega)\hat{\phi}(y)\rangle \end{pmatrix} \\ &= \begin{pmatrix} 1 & i\omega\rho(y) \\ -i\omega\rho(x) & \omega^2 \rho(x)\rho(y) \end{pmatrix} \tilde{F}(x,y,\omega), \end{aligned} \tag{6.6}$$

which can be expressed as

$$\tilde{\mathsf{F}}(x,y,\omega) = \sum_{jk} \tilde{a}_{jk}(\omega) \boldsymbol{f}_j(x) \otimes \boldsymbol{f}_k(y), \tag{6.7}$$

where $\tilde{a}_{jk}$ is evaluated to be (Appendix C)

$$\tilde{a}_{jk}(\omega) = \frac{\omega f_j(a) f_k(a)}{2(1-e^{-\beta\omega})\omega_j \omega_k (\omega_j - \omega)(\omega_k + \omega)}; \tag{6.8}$$

that is, the nondiagonal expansion (6.4) is the unique one that generalizes to the tensor $\tilde{\mathsf{F}}$ as in Eq. (6.7).

C. Density of states

Another important quantity is the density of states, which figures prominently in the heuristic argument of Purcell [4] and others [27]. The local density of states $d(x,\omega)$ [cf. Eq. (D3)], given below only for real positive ω, is related to the correlation function $\tilde{F}$ by

$$d(x,\omega) = -\frac{2\omega}{\pi} \mathrm{Im}\, \tilde{G}^{\mathrm{R}}(x,x,\omega) = \frac{\omega}{\pi}(1-e^{-\beta\omega})\tilde{F}(x,x,\omega), \tag{6.9}$$

which allows expression of this important quantity in terms of the QNMs. From Eq. (3.6) one gets

$$d(x,\omega) = \frac{\omega}{\pi} \sum_j \mathrm{Im} \frac{f_j^2(x)}{\omega_j(\omega_j - \omega)}, \tag{6.10}$$

while the nondiagonal expression (6.4) gives

$$d(x,\omega)=\frac{\omega^2}{2\pi}\sum_{jk}\ \frac{f_j(a)f_k(a)}{\omega_j\omega_k(\omega_j-\omega)(\omega_k+\omega)}f_j(x)f_k(x) \quad (6.11)$$

$$=\frac{1}{2\pi}\sum_{jk}\ \frac{f_j(a)f_k(a)}{(\omega-\omega_j)(\omega+\omega_k)}f_j(x)f_k(x). \quad (6.12)$$

The second form results from the first by use of the identity

$$\sum_j\ \frac{f_j(x)f_j(y)}{\omega_j}=0, \quad (6.13)$$

which follows from Eq. (2.7) by letting $t\downarrow 0$.

Superficially, the diagonal form is simpler. However, if we take a single resonance approximation, Eq. (6.10) yields, with one term j,

$$d(x,\omega)\approx\frac{\omega}{\pi}\,\mathrm{Im}\,\frac{f_j^2(x)}{\omega_j(\omega_j-\omega)}, \quad (6.14)$$

which is not positive definite. On the other hand, for the nondiagonal form, the appropriate approximation is to take one j and $k=-j$ in Eq. (6.12), leading to

$$d(x,\omega)\approx\frac{|f_j(a)f_j(x)|^2}{2\pi[(\omega-\mathrm{Re}\ \omega_j)^2+(\mathrm{Im}\ \omega_j)^2]}, \quad (6.15)$$

which is manifestly positive and, moreover, Lorentzian. From this expression one finds, to leading order in $|\mathrm{Im}\ \omega_j|=\gamma$, that

$$\int_{\mathrm{res}}d\omega\int_0^{a^+}dx\ \rho(x)d(x,\omega)\approx 1, \quad (6.16)$$

where the ω integral is over one resonance. This statement is readily derived from Eq. (6.15) by using Eq. (5.6) and the fact that $\int dx\ \rho(x)|f_j(x)|^2\approx 1$ for a narrow resonance. Recall that, in the modes of the universe approach [27], the unit weight of the resonances emerges simply as a numerical result, and is difficult to understand theoretically. Here the same result (in 1D) is justified analytically, and moreover, one can in principle (a) estimate the corrections due to other resonances (note that there is no ''background'' apart from the QNM contributions), (b) calculate the corrections to higher order in γ, and (c) discuss the local density of states $d(x,\omega)$ rather than the integrated $\int dx\ d(x,\omega)$. Incidentally, this discussion shows that of the two equivalent forms (6.11) and (6.12), the latter is the more appropriate, since it leads to a finite integral over ω in the single-resonance approximation.

One can derive another sum rule,

$$\int_0^{\omega}d(x,\omega')\ d\omega'\approx\frac{\omega}{\pi\sqrt{\rho(x)}}, \quad (6.17)$$

for large ω. This second sum rule [27] states that the states are merely redistributed without changing their total number. However, this sum rule is not immediately useful when expressed in terms of the QNMs, and will not be further discussed here.

VII. FEYNMAN PROPAGATOR

A. Derivation of the Feynman propagator

Another important correlation function is the Feynman propagator,

$$G^{\mathrm{F}}(x,y,t)=-i\langle\mathcal{T}\{\phi(x,t)\phi(y)\}\rangle, \quad (7.1)$$

in which $\mathcal{T}$ denotes time ordering. Taking the Fourier transform of the definition (7.1) leads to a direct relation to the correlator (3.4),

$$\tilde{G}^{\mathrm{F}}(x,y,\omega)=-\int\ \frac{d\omega'}{2\pi}\left\{\frac{1}{\omega'+\omega-i\varepsilon}+\frac{1}{\omega'-\omega-i\varepsilon}\right\}\times\tilde{F}(x,y,\omega'). \quad (7.2)$$

We shall limit the discussion below to $T=0$. Substitution of the right-hand side of Eq. (6.4) into Eq. (7.2) yields $\tilde{G}^{\mathrm{F}}$ as [28]

$$\tilde{G}^{\mathrm{F}}(x,y,\omega)=-i\sum_{jk}\ \frac{f_j(a)f_k(a)}{2\omega_j\omega_k(\omega_j+\omega_k)}\times\left\{\theta(\omega)\frac{\omega_j}{\omega_j-\omega}+\theta(-\omega)\frac{\omega_k}{\omega_k+\omega}\right\}f_j(x)f_k(y). \quad (7.3)$$

The cavity Feynman propagator can also be expressed in diagonal form, by substituting Eq. (3.6) into Eq. (7.2). Again taking $T=0$, this leads to

$$\tilde{G}^{\mathrm{F}}(x,y,\omega)=\frac{1}{2}\sum_j\ \frac{f_j(x)f_j(y)}{\omega_j(|\omega|-\omega_j)} \quad (7.4)$$

$$=\frac{1}{2}\sum_j\ \frac{f_j(x)f_j(y)}{|\omega|(|\omega|-\omega_j)}, \quad (7.5)$$

for real ω. It is stressed that these forms as single sums exist even though $\langle\mathcal{T}\{a_j(t)a_k\}\rangle\not\propto\delta_{j,\pm k}$ in general. The form (7.5) for $\tilde{G}^{\mathrm{F}}$ has been derived from Eq. (7.4) by means of the QNM identity (6.13). The second form with its divergence [29] at $\omega=0$ is less convenient than the first. It has been included to show that caution is needed when speaking about ''the contribution of one QNM.'' In fact the two summands are almost equal if $|\omega|\approx\omega_j$; such resonances are seen to be exclusively associated to terms with $j\geqslant 0$.

All of these equivalent expressions (7.3), (7.4), and (7.5) can be written generally as

$$\tilde{G}^{\mathrm{F}}(x,y,\omega)=\sum_{jk}\ f_j(x)\Delta_{jk}(\omega)f_k(y), \quad (7.6)$$

with different forms for Δ_{jk}. This has an obvious diagrammatic interpretation: the field at $x(y)$ couples to the QNM $j(k)$ with a vertex $f_j(x)(f_k(y))$, and the QNM propagates from mode j to mode k with an amplitude Δ_{jk}. This may be compared with the more familiar case of an infinite conservative system, say,

$$\tilde{G}^{\mathrm{F}}(x,y,\omega)=\int \frac{dp}{2\pi}\, e^{-ipx}\Delta(p,\omega)e^{ipy}. \tag{7.7}$$

It is seen that $\int dp\cdots$ is replaced by $\Sigma_{jk}\cdots$. An important goal of the present second-quantized theory is to study cavity-atom interactions [6], often referred to as CQED. The objective is to establish a set of ''QNM Feynman rules,'' in which each line in a diagram is represented not by a continuous momentum, but by one discrete index (or a pair of them)—not only for computational convenience, but also because each term can be associated with a cavity resonance. Such a discrete representation is especially useful for microscopic cavities, where the resonances are widely spaced in frequency. The above results are crucial for establishing these Feynman rules.

The possibility of alternate expressions for the propagator may recall a similar situation with gauge theories [30], though the reasons are quite different.

B. Decay rate and the resonance approximation

While the use of the Feynman propagators in an interacting theory will be presented elsewhere [6], it is nevertheless profitable at this point to consider the very simple example of an atom coupled to the field at a fixed point x; in the dipole approximation and for weak atom-field coupling, the decay rate is related to the equilibrium equal-space propagator

$$\tilde{D}(\omega)\equiv\tilde{G}^{\mathrm{F}}(x,x,\omega). \tag{7.8}$$

In particular, we shall be interested in the single-resonance approximation for $\tilde{D}$. The obvious choice is to take a single term of the sum in Eq. (7.4), i.e.,

$$\tilde{D}(\omega)\approx\tilde{D}_{\mathrm{ra}'}(\omega)\equiv\frac{f_j(x)^2}{2\omega_j(|\omega|-\omega_j)}. \tag{7.9}$$

The alternative is to start from Eq. (7.3), and retain only the $(j,-j)+(-j,j)$ terms (only when $k=-j$ does the factor $\omega_j+\omega_k$ in the denominator of Eq. (7.3) get small close to the conservative limit, which is the only case in which a single resonance can dominate) to arrive at

$$\tilde{D}(\omega)\approx\tilde{D}_{\mathrm{ra}}(\omega)\equiv\frac{|f_j(x)f_j(a)|^2}{4|\omega_j|^2|\mathrm{Im}\ \omega_j|}\left[\frac{\omega_j}{|\omega|-\omega_j}-\frac{\omega_j^*}{|\omega|+\omega_j^*}\right]. \tag{7.10}$$

Without loss of generality choosing $j>0$, the second (nonresonant) term in Eq. (7.10) is of the same order as those already neglected, and hence for most purposes may be omitted. However, only the sum of the two terms in Eq. (7.10) preserves the fundamental relation [20]

$$\tilde{D}^{\mathrm{R}}(\omega)=\tilde{D}^{\mathrm{A}*}(\omega) \tag{7.11}$$

for real ω, where $\tilde{D}^{\mathrm{R}}$ ($\tilde{D}^{\mathrm{A}}$) is the retarded (advanced) propagator obtained from $\tilde{D}(\omega)$ by continuation from positive (negative) frequencies. As a consequence, it turns out [6] that keeping both terms and using the ensuing cavity propagator to compute the self-energy of a two-level atom leads to a renormalization of the level splitting that is guaranteed to be real. Of course, for $\tilde{D}_{\mathrm{ra}'}$ the equality (7.11) is always violated.

Moreover, $\tilde{D}_{\mathrm{ra}'}$ does not obey the equally fundamental inequality $\mathrm{Im}\,\tilde{D}(\omega)\leqslant 0$ on the real axis [20] [see also (7.12) below], which $\tilde{D}_{\mathrm{ra}}$ satisfies term by term. Violation of this inequality in general leads to a retarded atom propagator that has poles in the upper half ω plane [6], signifying an unphysical instability.

To be sure, in spite of these crucial differences between $\tilde{D}_{\mathrm{ra}}$ and $\tilde{D}_{\mathrm{ra}'}$ their residues at $|\omega|=\omega_j$ agree in the conservative limit, in which the domination of a single QNM becomes rigorous. For a proof it suffices to note that $|f_j(a)|^2/2|\mathrm{Im}\ \omega_j|\to 1$ in this limit (Appendix A).

We have discussed the single-resonance approximation to both the density of states $d(x,\omega)$ and to the equal-space propagator $\tilde{D}(\omega)$. In fact, the arguments are equivalent, which can be appreciated physically from the fact that they both relate to the decay rate, and mathematically from the following identity for real positive ω:

$$d(x,\omega)=-\frac{2\omega}{\pi}\,\mathrm{Im}\,\tilde{D}(\omega). \tag{7.12}$$

In several places we have remarked that the nondiagonal QNM representation has some nice properties, and is in fact the unique representation if the field ϕ and the conjugate momentum $\hat{\phi}$ are considered together, for example, in the tensor correlator (6.7). There are of course many ways to understand why the correlator is nondiagonal; one of the most direct is via Eq. (4.4), which shows that all the mode coefficients a_j are driven by the same force $b(t)$, so in general different coefficients will have phase coherence and hence a nonzero correlation. Incidentally, this nondiagonal nature is *not* a quantum effect, since the property survives at high temperatures, e.g., $\beta\to 0$ in Eq. (6.4). However, all the propagators and correlation functions become diagonal in the conservative limit, as they should. In fact, applying Eq. (2.14) to Eq. (6.2) in this limit readily yields

$$\langle\tilde{a}_j(\omega)a_k\rangle=\frac{\pi}{\omega(1-e^{-\beta\omega})}\,\delta(\omega-\omega_j)\,\delta_{j,-k}, \tag{7.13}$$

in agreement with the creation-annihilation interpretation of the a_j in this limit given above Eq. (5.8). As a result both Eq. (6.4) for $\tilde{F}$ and Eq. (7.3) for $\tilde{G}^{\mathrm{F}}$ become diagonal in the conservative limit as well.

VIII. EXAMPLE: THE DIELECTRIC ROD

A useful check and example of the preceding is given by the ''dielectric rod'' model [11]:

$$\rho(x)=n^2\theta(a-x)+n_0^2\theta(x-a). \tag{8.1}$$

That is, we generalize the condition (2.6) and allow $\rho(x>a)$ to be an arbitrary constant n_0^2. To be sure, this generalization is trivial in principle since a model with param-

eters (n,n_0,a) can be mapped onto one with parameters $(n/n_0,1,n_0a)$ by the substitution $x\mapsto n_0x$. Yet it is convenient in practice, since we can now deal with two different conservative limits [see the discussion below Eq. (8.4)]: $n/n_0\to 0$ (the ''nodal'' limit) and $n/n_0\to\infty$ (the ''antinodal'' limit), by letting $n_0\to\infty$ and $n_0\to 0$, respectively, while keeping n, a, and hence Re ω_j [see Eq. (8.2) below] fixed.

The model (8.1) can be solved exactly for the QNM frequencies [12], which read [31]

$$na\omega_j=\begin{cases}(j+\frac{1}{2})\pi-i\,\mathrm{arcoth}(n/n_0), & n>n_0\\ j\pi-i\,\mathrm{artanh}(n/n_0), & n<n_0\end{cases} \tag{8.2}$$

$$=j\pi-\frac{i}{2}\ln\frac{n_0+n}{n_0-n}. \tag{8.3}$$

Both $n_0\to\infty$ and $n_0\to 0$ are indeed seen to be conservative limits (Im $\omega_j\to 0$). They correspond to clamped and free ends, respectively, in the interpretation of the wave equation as the transverse vibrations of a string [32,33]. On the other hand, for $n_0\to n$ the dissipation tends to infinity, and the QNM description breaks down in the limit $n_0=n$ (see, however, the end of this section).

The QNM wave functions are given inside the cavity by [31]

$$f_j(x)=\sqrt{\frac{2}{n^2a}}\sin(n\omega_jx); \tag{8.4}$$

their normalization is still given by Eq. (2.11), but our generalization $\rho(x>a)=n_0^2$ is readily shown to imply a corresponding modification of the surface term in the scalar product definition itself, viz.,

$$\langle\boldsymbol{\zeta},\boldsymbol{\chi}\rangle=i\left\{\int_0^{a^+}dx[\zeta(x)\hat{\chi}(x)+\hat{\zeta}(x)\chi(x)]+n_0\zeta(a)\chi(a)\right\}. \tag{8.5}$$

In the ''nodal'' limit $n_0\to\infty$, Eqs. (8.2) and (8.4) show that $f_j(a)\sim n_0^{-1}$, so that the factor $f_j(a)f_k(a)$ in the surface term of the orthogonality relation (2.14), (8.5) overcomes the explicit factor n_0, allowing the surface term to be neglected. This nodal limit has a counterpart in the ''loaded string'' model $\rho(x)=1+M\,\delta(x-a)$, where M can be set to infinity [34]. On the other hand, in the antinodal limit $n_0\to 0$ it is the explicit n_0 that allows neglect of the surface term, $f_j(a)$ tending to a constant. Hence, the QNM expansion becomes a standard normal-mode expansion also if $n_0\ll n$, which clarifies a detail left open in Sec. II and Ref. [12].

By means of a partial-fraction expansion and the identity (6.13), the expression (3.6) for $\tilde{F}$ can be rewritten as

$$\tilde{F}(x,y,\omega)=\frac{-i}{2\omega(1-e^{-\beta\omega})}\times\sum_j f_j(x)f_j(y)\left[\frac{1}{\omega_j-\omega}+\frac{1}{\omega_j+\omega}\right], \tag{8.6}$$

which is analytically more convenient even though the sum over j converges more slowly. Upon substitution of the dielectric rod QNMs [Eqs. (8.2) and (8.4)] into Eq. (8.6), the sum over j can be performed as the *conventional* Fourier series [35],

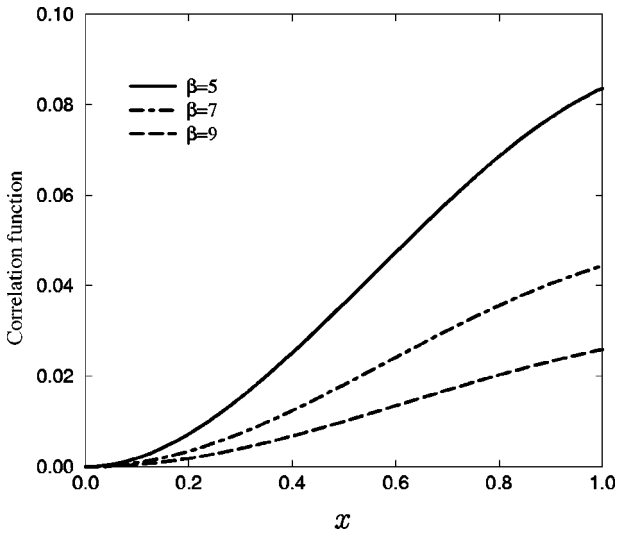

FIG. 1. Equal-space correlation function within the dielectric rod as a function of x at $t=0.1$ and different inverse temperatures β. The refractive indices are $n_0=1$, $n=5$; the width $a=1$.

$$\sum_j\frac{e^{-ji\pi z}}{j\pi-i\alpha}=\frac{2ie^{\alpha z}}{e^{2\alpha}-1},\quad 0<z<2 \tag{8.7}$$

implying

$$\sum_j\frac{e^{ji\pi z}}{j\pi-i\alpha}=\frac{2ie^{\alpha(2-z)}}{e^{2\alpha}-1},\quad 0<z<2, \tag{8.8}$$

and some rearrangement yields the correlation function as

$$\tilde{F}(x,y,\omega)=\frac{2n_0\sin(n\omega x)\sin(n\omega y)}{\omega(1-e^{-\beta\omega})[n_0^2\sin^2(n\omega a)+n^2\cos^2(n\omega a)]}. \tag{8.9}$$

This result also follows from the modes of the universe approach in Appendix D.

As discussed in Sec. III, the subtracted correlation function F_S is directly related to the energy density, and is in fact the squared amplitude of the field strength. Figure 1 shows $F_S(x,x,t)$ versus x at $t=0.1$ for the dielectric-rod model with $a=1$, $n_0=1$, $n=5$, for different values of β; this shows that the field amplitude is largest near the leaky end of the rod. Figure 2 shows $F_S(x,x,t)$ versus t at $x=0.3$ (all other parameters the same as before). This diagram vividly illustrates the advantage of the QNM approach—although the result is, in principle, obtainable from the modes of the universe method, the clear oscillatory signal is best captured by expressing this quantity in terms of QNMs.

Also for the Feynman propagator the sum (7.4) can be performed if $\rho=$ const, yielding

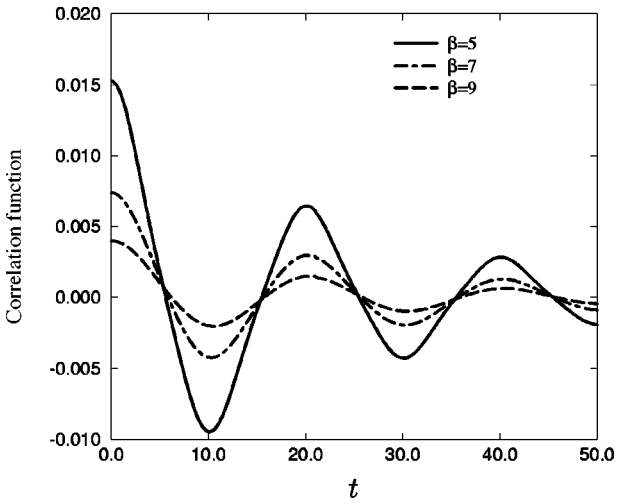

FIG. 2. Equal-space correlation function within the dielectric rod as a function of t at $x=0.3$ and different inverse temperatures β. The refractive indices are $n_0=1$, $n=5$; the width $a=1$.

$$\tilde{G}^{\mathrm{F}}(x,y,\omega)=-\frac{\sin(n\omega x)}{n\omega}\times\frac{n\cos[n\omega(a-y)]-in_0\sin[n|\omega|(a-y)]}{n\cos(n\omega a)-in_0\sin(n|\omega|a)} \tag{8.10}$$

for $x<y$, while for $x>y$ the propagator is obtained via $\tilde{G}^{\mathrm{F}}(x,y,\omega)=\tilde{G}^{\mathrm{F}}(y,x,\omega)$.

Notice that the final expressions (8.9) and (8.10) tend to a finite limit if $n_0\to n$ even though the individual terms in Eqs. (3.6) and (7.4) do not. This illustrates that the QNM expansion retains its validity up to arbitrarily large damping. In this semi-infinite string limit the very notions of cavity and environment lose their meaning, and indeed the right-hand sides of Eqs. (8.9) and (8.10) are seen to become independent of a.

IX. FINAL REMARKS

To summarize, we have developed the second-quantized version of the field theory using the QNM basis. Various physical quantities are then written as sums over QNM contributions—either as diagonal sums over a single index j, or as nondiagonal sums over a pair of indices jk. The resonance approximation is studied, leading to a proof of the unit weight of narrow resonances in the density of states, or, equivalently, the enhancement rate for the decay of excited states as embodied in the behavior of the equal-space propagator $\tilde{D}(\omega)$.

As has been mentioned already in Sec. I, an important extension of the present work is to include matter in the Hamiltonian (3.1), enabling the application of QNMs to quantum optics. This will be the subject of Ref. [6]. Other generalizations include the study of vector fields, and of open systems in three space dimensions. Further, a development parallelling the present one could be carried out for the Klein-Gordon equation instead of the wave equation (2.1) [9]. Since the two evolution equations are directly related by a transformation of the spatial variable [12], however, this has not been taken up here.

Non-Hermitian Hamiltonians and the ensuing complex eigenvalues also figure prominently in Siegman's work on dissipative CQED [36], which elaborates on that of Fox and Li [37], where already the latter considers eigenvalue problems for complex symmetric operators. However, [36] and [37] deal with transverse modes in the semiclassical limit [only considering (a) c-number fields with some effective quantum noise, and (b) the limit $\lambda\ll a$, where λ is the wavelength]. Thus the present paper pertains to a different regime, and there is the intriguing possibility of a future unifying approach.

Instead of generalizing the physical system one can also relax the assumption of global equilibrium made in Sec. VI. It is recalled here that the formalism of Secs. IV and V—and, in particular, the driving force b of Eq. (4.2)—is well defined for any initial state of the fields; taking a coherent state for the latter instead of a thermal one enables the study of a pumped cavity.

On the theoretical side, it would be interesting to provide a path-integral formulation of QNM quantization. This can supposedly be done on two levels. The first, semiphenomenological one is to write down an effective action generating dynamics equivalent to Eq. (4.4). The second, more fundamental one is to start with the action for the whole universe for our model (3.1), integrate out the degrees of freedom of the outside, and use a QNM basis for the ensuing dynamics of the cavity.

For such future developments, this paper can hopefully serve as a starting point and reference. In conclusion, we have shown that the QNM expansion is as powerful for open second-quantized systems as it is for their classical counterparts.

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APPENDIX A: RELATION BETWEEN SURFACE TERM AND IMAGINARY PART OF THE FREQUENCY

In this appendix we give an alternative proof of the identity (5.6) in the conservative limit. For this purpose, generalize to complex classical fields and define the energy density $h(x)=|\partial_x\phi|^2/2+|\hat{\phi}|^2/2\rho(x)$, so that $\dot{h}(x)=-\partial_x j(x)$ with the current $j(x)=-\mathrm{Re}[\hat{\phi}(x)\partial_x\phi(x)]/\rho(x)$. Define the cavity energy $E=\int_0^{a^+}dx\,h(x)$, then for a field $\phi(x,t)=f_j(x)e^{-i\omega_j t}$ at $t=0$ one has

$$E=\frac{\gamma}{2}|f_j(a)|^2+(\mathrm{Re}\ \omega_j)^2\int_0^{a^+}dx\ \rho(x)|f_j(x)|^2, \tag{A1}$$

where $\gamma\equiv|\mathrm{Im}\ \omega_j|$. In the conservative limit the first term vanishes, while the integral in the second term tends to unity so that $E\to|\omega_j|^2$. Combination with $-\dot{E}=2\gamma E=j(a^+)=|\omega_j|^2|f_j(a)|^2$ shows that $|f_j(a)|^2/2\gamma\to 1$ in this limit, which proves our assertion.

APPENDIX B: IDENTITY FOR RETARDED PROPAGATOR

In this appendix we derive the Green's-function identity (6.5). To this end, we define $f(x,\omega)$ $(g(y,\omega))$ as the solution of the homogeneous wave equation (2.4) (upon the substitution $\omega_j \mapsto \omega$) satisfying the first (second) of the boundary conditions (2.5) [38]. This allows one to write

$$\tilde{G}^{\mathrm{R}}(x<y,\omega)=\frac{f(x,\omega)g(y,\omega)}{W(\omega)}, \tag{B1}$$

where one can choose $f(x,\omega)=f(x,-\omega)=f^*(x,\omega)$ and $g(y,\omega)=g^*(y,-\omega)$, and where W is the position-independent Wronskian of the functions f and g [11]. Then one has

$$\begin{aligned}\frac{\tilde{G}^{\mathrm{R}}(x,y,\omega)-\tilde{G}^{\mathrm{R}}(x,y,-\omega)}{\tilde{G}^{\mathrm{R}}(x,a,\omega)\tilde{G}^{\mathrm{R}}(y,a,-\omega)}&=\frac{g(y,\omega)W^*(\omega)-g^*(y,\omega)W(\omega)}{|g(a,\omega)|^2 f(y,\omega)}\\&=\frac{g(y,\omega)g'^*(y,\omega)-g'(y,\omega)g^*(y,\omega)}{|g(a,\omega)|^2}.\end{aligned} \tag{B2}$$

The numerator of this last expression is itself another Wronskian, and hence can be evaluated at $y=a^+$ to yield

$$\begin{aligned}\frac{\tilde{G}^{\mathrm{R}}(x,y,\omega)-\tilde{G}^{\mathrm{R}}(x,y,-\omega)}{\tilde{G}^{\mathrm{R}}(x,a,\omega)\tilde{G}^{\mathrm{R}}(y,a,-\omega)}&=-2i\ \mathrm{Im}\frac{g'(a^+,\omega)}{g(a,\omega)}\\&=\frac{2\omega}{i},\end{aligned} \tag{B3}$$

completing the proof of Eq. (6.5).

APPENDIX C: EXPANSION OF TENSOR CORRELATOR

The coefficients $\tilde{a}_{jk}$ in Eq. (6.7) are given by the projection

$$\tilde{a}_{jk}(\omega)=\frac{\langle\langle \tilde{\mathsf{F}}(\omega), \boldsymbol{f}_j\otimes\boldsymbol{f}_k\rangle\rangle}{4\omega_j\omega_k} \tag{C1}$$

in terms of the bilinear form on the product space, which reads

$$\begin{aligned}\langle\langle \mathsf{P},\mathsf{Q}\rangle\rangle\equiv&-\int_0^{a^+}dx\ dy\{P_{11}(x,y)Q_{22}(x,y)+P_{12}(x,y)Q_{21}(x,y)+P_{21}(x,y)Q_{12}(x,y)+P_{22}(x,y)Q_{11}(x,y)\}\\&-\int_0^{a^+}dx\{P_{11}(x,a)Q_{21}(x,a)+P_{21}(x,a)Q_{11}(x,a)\}-\int_0^{a^+}dy\{P_{11}(a,y)Q_{12}(a,y)+P_{12}(a,y)Q_{11}(a,y)\}\\&-P_{11}(a,a)Q_{11}(a,a).\end{aligned} \tag{C2}$$

Substitution of Eq. (6.6) for $\tilde{\mathsf{F}}(\omega)$ into Eq. (C1) yields

$$\begin{aligned}\tilde{a}_{jk}(\omega)=\frac{1}{4\omega_j\omega_k}\Bigg\{&(\omega_j+\omega)(\omega_k-\omega)\int_0^{a^+}dx\ dy\ \rho(x)\rho(y)f_j(x)f_k(y)\tilde{F}(x,y,\omega)+i(\omega_j+\omega)f_k(a)\\&\times\int_0^{a^+}dx\ \rho(x)f_j(x)\tilde{F}(x,a,\omega)+i(\omega_k-\omega)f_j(a)\int_0^{a^+}dy\ \rho(y)f_k(y)\tilde{F}(a,y,\omega)-f_j(a)f_k(a)\tilde{F}(a,a,\omega)\Bigg\}.\end{aligned} \tag{C3}$$

Inserting *any* QNM expansion $\tilde{F}(x,y,\omega)=\Sigma_{lm}\tilde{b}_{lm}(\omega)f_l(x)f_m(y)$ such as Eq. (3.6) or Eq. (6.4) and invoking the relations (2.11), (2.14), and (6.13), the expression (C3) can be evaluated as in Eq. (6.8), which is what we set out to show.

APPENDIX D: MODES OF THE UNIVERSE APPROACH TO THE CORRELATION FUNCTION

It is instructive to rederive the correlator F using the modes of the universe (MU). The MU expansion of the fields reads [39]

$$\begin{pmatrix}\phi(x)\\ \hat{\phi}(x)\end{pmatrix}=\sum_{l>0}\begin{pmatrix}(u_l^\dagger+u_l)/\sqrt{2\nu_l}\\ i\rho\sqrt{\nu_l/2}(u_l^\dagger-u_l)\end{pmatrix}\psi(x,\nu_l), \tag{D1}$$

i.e., it is of the same form as Eq. (5.8) but in Eq. (D1) the sum runs over the MU frequencies $\nu_l=l\pi/\Lambda$ (to leading order in $a/\Lambda\ll 1$), the u_l and ψ_l being MU annihilation operators and wave functions, respectively. Insertion of Eq. (D1) into Eq. (3.4) yields

$$\tilde{F}(x,y,\omega)=\sum_{lm}\frac{\psi(x,\nu_l)\psi(y,\nu_m)}{2\sqrt{\omega_l\omega_m}}\times\langle\{\tilde{u}_l^\dagger(\omega)+\tilde{u}_l(\omega)\}\{u_m^\dagger+u_m\}\rangle$$
$$=\sum_{l}\frac{\psi(x,\nu_l)\psi(y,\nu_l)}{2\omega_l}\langle\tilde{u}_l^\dagger(\omega)u_l+\tilde{u}_l(\omega)u_l^\dagger\rangle$$
$$=\frac{\Lambda}{|\omega|}\psi(x,|\omega|)\psi(y,|\omega|)\times\{\theta(-\omega)N(-\omega)+\theta(\omega)[N(\omega)+1]\}$$
$$=\frac{\Lambda\psi(x,|\omega|)\psi(y,|\omega|)}{\omega(1-e^{-\beta\omega})}. \quad \text{(D2)}$$

To arrive at the third line, we used $\tilde{u}_l^{(\dagger)}(\omega)=2\pi\times\delta(\omega\pm\nu_l)u_l^{(\dagger)}$ [upper sign for $\tilde{u}_l^\dagger(\omega)$] and defined the boson occupation number $N(\omega)=[\exp(\beta\omega)-1]^{-1}$. Comparison of Eqs. (6.4) and (D2) elucidates why the former factorizes with respect to x and y: this is seen to be a consequence of the nondegeneracy of the MU spectrum of the semi-infinite string, as opposed to, e.g., a free string or one with periodic boundary conditions. Evaluation of ψ for the ''dielectric rod'' of Sec. VIII at once shows that Eq. (D2) indeed reduces to Eq. (8.9) in this case, providing a further comparison among the various techniques of this paper.

Finally, in terms of the MU the local density of states is defined as

$$d(x,\omega)\equiv\sum_l|\psi(x,\nu_l)|^2\delta(\nu_l-\omega)=\frac{\Lambda}{\pi}\psi(x,\omega)^2, \quad \text{(D3)}$$

and comparison with Eq. (D2) at once reproduces Eq. (6.9).

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[8] See, e.g., S. Chandrasekhar, *The Mathematical Theory of Black Holes* (University of Chicago Press, Chicago, 1991).

[9] E. S. C. Ching, P. T. Leung, W. M. Suen, and K. Young, Phys. Rev. Lett. **74**, 2414 (1995); **74**, 4588 (1995); Phys. Rev. D **52**, 2118 (1995); **54**, 3778 (1996).

[10] Since the positivity of ρ limits its singularity at $x=a$ to at most a δ function, ϕ itself is continuous. Stronger singularities in ρ would also leave equations such as Eq. (2.4) undefined distributionally.

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[14] The technical condition that ϕ and $\hat{\phi}/\rho$ be differentiable, as well as the speed of convergence of the sum in Eq. (2.8), are outside the present scope; see Ref. [12].

[15] Notice that, while novel for open classical systems as in Eq. (2.8), the concept of two-component expansion—where the same $\alpha_j^{(\dagger)}$ are used to expand *both* ϕ *and* $\hat{\phi}$ as in Eq. (5.8)—enjoys a wide textbook popularity for closed quantum systems. See, e.g., Ref. [16].

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[22] The fact that the sum in Eq. (2.8) converges more slowly if $\boldsymbol{\phi}\notin\Gamma$ [12] is irrelevant in the present formal development. In any case, for operator fields the large-wave-number components never get small as they do for smooth classical fields, irrespective of the expansion method. For numerical applications, it is best to optimize the final formulas for observable physical quantities for convergence along the lines of Sec. III of Ref. [13]. Notice, however, that comparison of methods *B*

and C in Ref. [12] shows the surface term in Eq. (4.3) to improve convergence also for function pairs outside Γ, in particular, to be essential for convergence to the second component.

[23] However, we shall not notationally distinguish between corresponding classical and quantized fields, since the meaning will be clear from the context.

[24] Notice that *in general* $a_{-j}=a_j^\dagger$ provided that one chooses $f_{-j}=f_j^*$, a relation that is imposed by normalization only up to a sign.

[25] For simplicity of notation we discuss Eq. (6.5) for real ω only, but its proof can be extended to complex ω by continuing both sides analytically.

[26] This expression is actually symmetric under $x\leftrightarrow y$, as will be shown in Appendix D.

[27] S. C. Ching, H. M. Lai, and K. Young, J. Opt. Soc. Am. B **4**, 1995 (1987); **4**, 2004 (1987).

[28] Equations (7.3), (7.4), (7.9), and (7.10) apply for real ω, and the usual care [20] is needed when performing an analytic continuation from either positive or negative frequencies.

[29] Per term; after summation over n the factor multiplying $|\omega|^{-1}$ is seen to vanish near $\omega=0$, as required by consistency.

[30] See, e.g., E. S. Abers and B. W. Lee, Phys. Rep. **9**, 1 (1973).

[31] In the case $n>n_0$ our numbering violates the convention $\omega_{-j}=-\omega_j^*$. Likewise Eq. (8.4) violates the sign convention given earlier. However, these cause no problems here.

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[34] This corresponds to a model of an optical cavity with a thin slab of material of high dielectric constant, forming a mirror with some output coupling; see, e.g., Ref. [1].

[35] Of course the projection formula for the latter, as opposed to the one for a QNM expansion, involves complex conjugation.

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[38] Up to normalization $f(x,\omega)=\psi(x,\omega)$, the MU wave function ψ of Appendix D being defined on a continuum of frequencies in the limit $\Lambda\to\infty$.

[39] In Eq. (5.8), the normalization (2.11) forces the f_j to be real in the conservative limit, in contrast to the standard formalism where their phase is arbitrary. For uniformity we choose the ψ in Eq. (D1) to be real as well; this shows that the result in Eq. (D2) is well defined.